

LDPC codes in the McEliece cryptosystem: attacks and countermeasures

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Abstract. The McEliece cryptosystem is a public-key cryptosystem based on coding theory that has successfully resisted cryptanalysis for thirty years. The original version, based on Goppa codes, is able to guarantee a high level of security, and is faster than competing solutions, like RSA. Despite this, it has been rarely considered in practical applications, due to two major drawbacks: i) large size of the public key and ii) low transmission rate. Several attempts have been made for overcoming such drawbacks, but the adoption of most families of codes has not been possible without compromising the system security. Low-Density Parity-Check (LDPC) codes are state-of-art forward error correcting codes that permit to approach the Shannon limit while ensuring limited complexity. Quasi-Cyclic (QC) LDPC codes are a particular class of LDPC codes, able to join low complexity encoding of QC codes with high-performing and low-complexity decoding techniques based on the belief propagation principle. In a previous work it has been proposed to adopt a particular family of QC-LDPC codes in the McEliece cryptosystem to reduce the key size and increase the transmission rate. It has been shown that such variant is able to counter all the classic attacks, and also attacks that can compromise the security of previous LDPC-based versions. Recently, however, new attacks have been found that are able to exploit a flaw in the transformation from the private key to the public one. Such attacks can be effectively countered by changing the form of some constituent matrices, without altering the system parameters. This change has marginal effects on the complexity of the cryptosystem that, instead, preserves its security against all known attacks. This work gives an overview of the QC-LDPC codes-based McEliece cryptosystem and its cryptanalysis. Two recent versions are considered, and their ability to counter all the currently known attacks is discussed. A third version able to reach a higher security level is also proposed. Finally, it is shown that the new QC-LDPC codes-based cryptosystem scales favorably when larger keys are needed, as very recently pointed out by the successful implementation of an attack against the original cryptosystem.

Keywords. McEliece Cryptosystem, LDPC Codes, Cryptanalysis.

Introduction

First presented by Robert J. McEliece in 1978 [1], the McEliece cryptosystem represents one of the most famous examples of error correcting codes-based public key cryptosys-

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tem. It adopts generator matrices of linear block codes as private and public keys, and the combination of a dense transformation and a permutation to hide the structure of the secret code into the public generator matrix. Its security lies in the difficulty of decoding a large linear code having no visible structure, that is an NP complete problem [2]. The McEliece cryptosystem has successfully resisted cryptanalysis for thirty years, and no algorithm able to realize a total break in a reasonable time has been found up to now.

Attacks achieving the lowest work factors aim at solving the general decoding problem, that consists in deriving the error vector affecting a codeword of an (n, k) -linear block code (*i.e.*, having length n and dimension k). It can be shown that this problem can be translated into that of finding the minimum weight codeword in an $(n, k + 1)$ -linear block code, so the McEliece cryptosystem can also be attacked by means of algorithms aimed at finding low weight codewords.

A first decoding attack was already proposed by McEliece in his paper [1] and is based on the principle of *information set decoding*. It consists in selecting k bits of the ciphertext and inverting the encoding map, hoping that none of them is in error. This attack has been further improved by Lee and Brickell [3], who proposed a systematic procedure for validating the decoded words and showed that the attack can be attempted also when the chosen information set is affected by a small number of errors.

More recent decoding attacks are instead based on probabilistic algorithms searching for low weight codewords. Stern's algorithm [4] is among the most famous ones, and it has been later improved by Canteaut and Chabaud [5]. Very recently, Bernstein et al. have proposed a highly efficient implementation of the attack based on Stern's algorithm [6], that is able to achieve a speedup of about 12. The improved algorithm has been run on a computer cluster, and an encrypted codeword of the original McEliece cryptosystem has been correctly deciphered, thus proving the feasibility of an attack for the original choice of the system parameters.

Despite this, no polynomial time attack has been found up to now, and the system remains secure, provided that large enough keys are adopted in order to reach suitable work factors on modern computers. In addition, the McEliece cryptosystem can be considered to be a *post-quantum* cryptographic system [7], since no polynomial time algorithm able to exploit quantum computers for an attack has been found up to now. On the contrary, Shor presented a quantum polynomial time algorithm for calculating discrete logarithms that should be able to break RSA, DSA and ECDSA [8].

Moreover, the original version of the McEliece cryptosystem, based on binary Goppa codes with irreducible generator polynomials, can be two or three orders of magnitude faster than RSA. However, unlike RSA, the original McEliece cryptosystem has been rarely considered in practical applications, due to its two major drawbacks: large keys and low transmission rates. Many attempts have been made for replacing Goppa codes with other families of codes in order to overcome such drawbacks, but they always compromised the system security. This occurred for Generalized Reed-Solomon Codes [9] and Reed-Muller codes [10]. Successful total break attacks have also been conceived for some versions adopting Quasi-Cyclic (QC) codes [11] and Low-Density Parity-Check (LDPC) codes [12,13].

LDPC codes represent the state of the art in forward error correction and are able to approach the ultimate capacity bounds [14]. Their performance under belief propagation decoding depends on the characteristics of their sparse parity-check matrices and their design can be performed on a random basis. Thus, it is possible to obtain large families

of equivalent codes, that is the first requisite for their application in cryptography. The adoption of LDPC codes in the McEliece cryptosystem can yield many advantages: the sparse nature of their parity-check matrices could help to reduce the key size, at least in principle, and their easy design could allow to increase the transmission rate. Unfortunately, the usage of LDPC matrices as public keys can compromise the system security [12,13,15]. For this reason, it has been proposed to adopt public keys in the form of generator matrices of a particular family of QC-LDPC codes, that are structured LDPC codes. Their structured character allows to reduce the key size though using dense generator matrices.

Even with this choice, the adoption of sparse and block-wise diagonal transformation matrices can still expose the cryptosystem to total break attacks [16]; so, the original proposal has been recently revised in such a way to not include this kind of matrices. The new cryptosystem is immune against all currently know attacks, it allows a significant reduction in the key size with respect to the original version and achieves increased transmission rate. Furthermore, the size of its public keys increases linearly with the code dimension; so the new cryptosystem scales favorably when larger keys are needed for facing the growing computational power of modern computers.

The paper is organized as follows: Section 1 describes the original McEliece cryptosystem, while Section 2 is focused on its variants based on LDPC codes. In Section 3 the most dangerous attacks against the cryptosystem security are studied, together with their possible countermeasures. Section 4 is devoted to the complexity assessment of the considered cryptosystems and Section 5 concludes the paper.

1. The original McEliece cryptosystem

Inspired by the introduction of asymmetric cryptography by Diffie and Hellmann [17], McEliece proposed his code-based public key cryptosystem starting from the observation that a fast decoding algorithm exists for a general Goppa code, while the same does not occur for a general linear code [1].

In the McEliece cryptosystem, Bob randomly chooses an irreducible polynomial of degree t over $GF(2^m)$, that corresponds to an irreducible Goppa code of length $n = 2^m$ and dimension $k \geq n - tm$, able to correct t or fewer errors in each codeword. Then, Bob produces a $k \times n$ generator matrix \mathbf{G} for the secret code, in reduced echelon form, that will be part of his secret key. The remaining part of the secret key is formed by two other matrices: a dense $k \times k$ non singular matrix \mathbf{S} and a random $n \times n$ permutation matrix \mathbf{P} .

Then, Bob produces his public key as follows (the inverses of \mathbf{S} and \mathbf{P} are used here, rather than in the decryption map, for consistency with the notation used for the new proposals):

$$\mathbf{G}' = \mathbf{S}^{-1} \cdot \mathbf{G} \cdot \mathbf{P}^{-1}. \quad (1)$$

Alice, in order to send encrypted messages to Bob, fetches his public key \mathbf{G}' from the public directory, divides her message into k -bit words, and applies the encryption map as follows:

$$\mathbf{x} = \mathbf{u} \cdot \mathbf{G}' + \mathbf{e}, \quad (2)$$

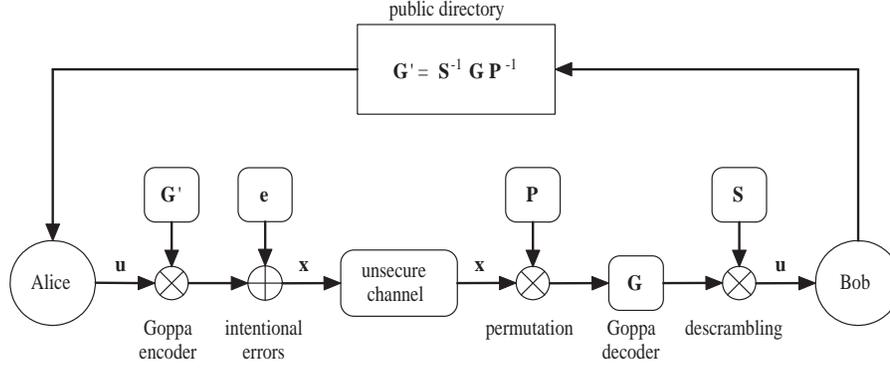


Figure 1. The original McEliece cryptosystem.

where x is the ciphertext corresponding to the cleartext u and e is a random vector of t intentional errors.

Bob, after having received the encrypted message x , inverts the secret permutation, thus finding a codeword of the secret Goppa code affected by the vector of intentional errors $e \cdot P$, having weight t :

$$x' = x \cdot P = u \cdot S^{-1} \cdot G + e \cdot P. \quad (3)$$

By exploiting Goppa decoding, Bob is able to correct all the t intentional errors. Hence he can obtain $u \cdot S^{-1}$, due to the systematic form of G , and then recover u through multiplication by S . The main blocks of the McEliece cryptosystem are shown in Figure 1.

In his original formulation, McEliece adopted Goppa codes with length $n = 1024$ and dimension $k = 524$, able to correct up to $t = 50$ errors. The key size is hence $n \times k = 67072$ bytes, and the transmission rate is $k/n \approx 0.5$. On the other hand, the RSA system with 1024-bit modulus and public exponent 17 has keys of just 256 bytes and reaches unitary transmission rate (*i.e.*, encryption has no overhead on the transmission).

However, it must be considered that the McEliece cryptosystem is significantly faster than RSA: it requires 514 binary operations per bit for encoding and 5140 for decoding. On the contrary, RSA requires 2402 and 738112 binary operations per bit for encoding and decoding, respectively [5].

2. LDPC codes in the McEliece cryptosystem

In this section a recent version of the McEliece cryptosystem based on QC-LDPC codes is described. It exploits the peculiarities of QC-LDPC codes for overcoming the drawbacks of the original system and it is able to resist all attacks currently known.

First, some basic properties of QC-LDPC codes are reminded, then it is shown how the McEliece cryptosystem should be modified in order to use these codes as private and public keys without incurring in security issues.

2.1. QC-LDPC codes based on difference families

LDPC codes represent a particular class of linear block codes, able to approach channel capacity when soft decision decoding algorithms based on the *belief propagation* principle are adopted [14].

An (n, k) LDPC code C is defined as the kernel of a sparse $(n - k) \times n$ parity-check matrix \mathbf{H} :

$$C = \{ \mathbf{c} \in GF(2)^n : \mathbf{H} \cdot \mathbf{c}^T = \mathbf{0} \}. \quad (4)$$

In order to achieve very good performance under belief propagation decoding, the parity-check matrix \mathbf{H} must have a low density of 1 symbols (typically on the order of 10^{-3}) and absence of short cycles in the associated Tanner graph. The shortest possible cycles, that have length four, are avoided when any pair of rows (columns) has supports with no more than one overlapping position.

These conditions suffice to obtain good LDPC codes; so they can be designed through algorithms that work directly on the parity-check matrix, aiming at maximizing the cycles length, like the Progressive Edge Growth (PEG) algorithm [18]. The codes obtained are unstructured, in the sense that the positions of 1 symbols in each row (or column) of the parity-check matrix are independent of the others. This feature influences complexity of the encoding and decoding stages, since the whole matrix must be stored and the codec implementation cannot take advantage of any cyclic or polynomial nature of the code. In this case, a common solution consists in adopting lower triangular or quasi-lower triangular parity-check matrices, that correspond to sparse generator matrices, in such a way as to reduce complexity of the encoding stage [19].

Opposite to this approach, structured LDPC codes have also been proposed, whose parity-check matrices have a very simple inner structure. Among them, QC-LDPC codes represent a very important class, able to join easy encoding of QC codes with the astonishing performance of LDPC codes. For this reason, QC-LDPC codes have been included in several recent telecommunication standards and applications [20,21].

QC-LDPC codes have both length and dimension multiple of an integer p , that is, $n = n_0 p$ and $k = k_0 p$. They have the property that each cyclic shift of a codeword by n_0 positions is still a valid codeword. This reflects on their parity-check matrices, that are formed by circulant blocks. A $p \times p$ circulant matrix \mathbf{A} over $GF(2)$ is defined as follows:

$$\mathbf{A} = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{p-1} \\ a_{p-1} & a_0 & a_1 & \cdots & a_{p-2} \\ a_{p-2} & a_{p-1} & a_0 & \cdots & a_{p-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_0 \end{bmatrix}, \quad (5)$$

where $a_i \in GF(2)$, $i = 0 \dots p - 1$.

A simple isomorphism exists between the algebra of $p \times p$ binary circulant matrices and the ring of polynomials $GF(2)[x]/(x^p + 1)$. If we denote by \mathbf{X} the unitary cyclic permutation matrix, the isomorphism maps \mathbf{X} into the monomial x and the circulant matrix $\sum_{i=0}^{p-1} \alpha_i \mathbf{X}^i$ into the polynomial $\sum_{i=0}^{p-1} \alpha_i x^i \in GF(2)[x]/(x^p + 1)$. This isomorphism can be easily extended to matrices formed by circulant blocks.

Let us focus attention on a particular family of QC-LDPC codes, having the parity-check matrix formed by a single row of n_0 circulant blocks, each with row (column) weight d_v :

$$\mathbf{H} = [\mathbf{H}_0 | \mathbf{H}_1 | \dots | \mathbf{H}_{n_0-1}]. \quad (6)$$

If we suppose (without loss of generality) that \mathbf{H}_{n_0-1} is non singular, a valid generator matrix for the code in systematic form can be expressed as follows:

$$\mathbf{G} = \mathbf{I} \begin{bmatrix} (\mathbf{H}_{n_0-1}^{-1} \cdot \mathbf{H}_0)^T \\ (\mathbf{H}_{n_0-1}^{-1} \cdot \mathbf{H}_1)^T \\ \vdots \\ (\mathbf{H}_{n_0-1}^{-1} \cdot \mathbf{H}_{n_0-2})^T \end{bmatrix}, \quad (7)$$

where \mathbf{I} represents the $k \times k$ identity matrix.

Very simple methods for designing parity-check matrices in the form (6), free of length-4 cycles, are those exploiting difference families and their variants [22,23,24]. Such methods are based on the observation that, if we denote as $\mathbf{h}_i, i = 0 \dots n_0 - 1$, the vector containing the positions of 1 symbols in the first row of \mathbf{H}_i , the absence of length-4 cycles in \mathbf{H} is ensured when all the \mathbf{h}_i 's have disjoint sets of differences modulo p . Sets of \mathbf{h}_i 's with such property can be obtained on a random basis, so yielding large families of codes with identical parameters [13].

All the codes in a family share the characteristics that mostly influence performance of belief propagation decoding, that are: code length and dimension, parity-check matrix density, nodes degree distributions and cycles length distribution. So, they have equivalent error correction performance under belief propagation decoding.

In order to apply such codes within the framework of the McEliece cryptosystem, it is interesting to assess their error correction capability over a channel that adds exactly t errors in each codeword. This channel can be seen as a variant of the Binary Symmetric Channel (BSC), and will be denoted as the *McEliece channel* in the following. This evaluation can be done through numerical simulations: Figure 2 shows the performance in terms of Bit Error Rate (BER) and Frame Error Rate (FER) of three QC-LDPC codes that will be of interest in the following. They have $(n, k) = (16384, 12288)$, $(24576, 16384)$ and $(49152, 32768)$, respectively.

It is important to note that the decoding radius of LDPC codes over the McEliece channel cannot be determined analytically, as instead occurs for Goppa codes; so, we can only choose values of t that are able to ensure an extremely low error rate.

2.2. McEliece cryptosystem adopting QC-LDPC codes

The adoption of QC-LDPC codes in the McEliece cryptosystem can yield important advantages in terms of key size and transmission rate. As any other family of linear block codes, QC-LDPC codes are exposed to the same attacks targeted to the original cryptosystem; among them, decoding attacks represent the most dangerous ones (as it will be shown in Section 3.3).

Moreover, the adoption of LDPC codes could expose the system to new attacks, due to the sparse nature of their matrices. It was already observed in [12] that LDPC

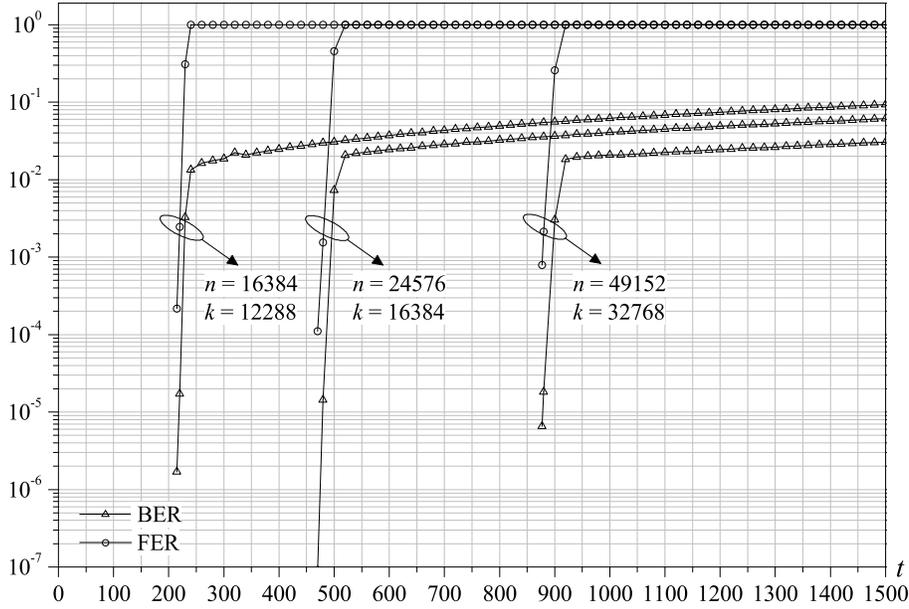


Figure 2. Performance of some QC-LDPC codes over the McEliece channel.

matrices cannot be used for obtaining the public key, not even after applying a linear transformation through a sparse matrix. In this case, the secret LDPC matrix could be recovered through *density reduction attacks*, that aim at finding the rows of the secret matrix by exploiting their low density [12,25].

One could think to replace LDPC matrices with their corresponding generator matrices that, in general, are dense. Actually, this is what happens in the original McEliece cryptosystem, where a systematic generator matrix for the secret Goppa code is used, hidden through a permutation. However, a permutationally equivalent code of an LDPC code is still an LDPC code, and the rows of its LDPC matrix could be found by searching for low weight codewords in the dual of the secret code. We call this strategy *attack to the dual code*: it aims at finding a sparse representation for the parity-check matrix of the public code, that can be used for effective LDPC decoding.

So, when adopting LDPC codes in the McEliece cryptosystem, it does not suffice to hide the secret code through a permutation, but it must be ensured that the public code does not admit sparse characteristic matrices. For this reason, it has been proposed to replace the permutation matrix \mathbf{P} with a different transformation matrix, \mathbf{Q} [13]. \mathbf{Q} is a sparse $n \times n$ matrix, with rows and columns having Hamming weight $m > 1$. This way, the LDPC matrix of the secret code (\mathbf{H}) is mapped into a new parity-check matrix that is valid for the public code:

$$\mathbf{H}' = \mathbf{H} \cdot \mathbf{Q}^T. \quad (8)$$

Depending on the value of m , the density of \mathbf{H}' could be rendered high enough to avoid attacks to the dual code.

Table 1. Choices of the parameters for the QC-LDPC-based McEliece cryptosystem.

System	n_0	d_v	p	m	t'	Key size (bytes)
1	4	13	4096	7	27	6144
2	3	13	8192	11	40	6144
3	3	15	16384	13	60	12288

In the modified cryptosystem, Bob chooses a secret LDPC code by fixing its parity-check matrix, \mathbf{H} , and selects two other secret matrices: a $k \times k$ non singular scrambling matrix \mathbf{S} and an $n \times n$ non singular transformation matrix \mathbf{Q} with row/column weight m . Then, Bob obtains a systematic generator matrix \mathbf{G} for the secret code and produces his public key as follows:

$$\mathbf{G}' = \mathbf{S}^{-1} \cdot \mathbf{G} \cdot \mathbf{Q}^{-1}. \quad (9)$$

It should be noted that the public key is a dense matrix, so the sparse character of LDPC codes does not help reducing the key length. However, when adopting QC-LDPC codes, the characteristic matrices are formed by circulant blocks that are completely described by a single row or column. This fact significantly reduces the key length that, moreover, increases linearly with the code length.

The encryption map is the same as in the original cryptosystem: \mathbf{G}' is used for encoding and a vector \mathbf{e} of intentional errors is added to the encoded word. The Hamming weight of vector \mathbf{e} , in this case, is denoted as t' . The decryption map must be slightly modified with respect to the original cryptosystem. After having received a ciphertext, Bob must invert the transformation as follows:

$$\mathbf{x}' = \mathbf{x} \cdot \mathbf{Q} = \mathbf{u} \cdot \mathbf{S}^{-1} \cdot \mathbf{G} + \mathbf{e} \cdot \mathbf{Q}, \quad (10)$$

thus obtaining a codeword of the secret LDPC code affected by the error vector $\mathbf{e} \cdot \mathbf{Q}$ with weight $\leq t = t'm$. After that, Bob must be able to correct all the errors through LDPC decoding and obtain $\mathbf{u} \cdot \mathbf{S}^{-1}$, due to the systematic form of \mathbf{G} . Finally, he can recover \mathbf{u} through multiplication by \mathbf{S} .

It should be noted that the introduction of the transformation matrix \mathbf{Q} in place of the permutation matrix causes an error amplification effect (by a factor m). This is compensated by the error correction capability of the secret LDPC code, that must be able to correct t errors.

Based on this scheme, two possible choices of the system parameters have been recently proposed, that are able to ensure different levels of security against currently known attacks [26]. A third choice is here considered that demonstrates how the cryptosystem scales favorably when larger keys are needed for facing efficient implementations of the attacks, as the one proposed recently. For the three codes considered (whose performance is reported in Figure 2), $t = 189, 440$ and 780 has been assumed, respectively, and m and t' have been fixed accordingly. The considered values of the parameters are summarized in Table 1. It should be noted that the key size is simply $k_0 n_0 p$, since the whole matrix can be described by storing only the first row (or column) of each circulant block.

Table 2. Work factors of attacks to the dual code.

System	n_0	d_v	p	m	Max WF	$w(\text{WF} \geq 2^{80})$
1	4	13	4096	7	2^{153}	179
2	3	13	8192	11	2^{250}	127
3	3	15	16384	13	2^{340}	124

3. Attacks and countermeasures

For the sake of conciseness, this section considers only the attacks that are able to achieve the lowest work factors for the considered cryptosystem, together with their possible countermeasures.

3.1. Attacks to the dual code

This kind of attacks exploits the fact that the dual of the public code, that is generated by \mathbf{H}' , may contain low weight codewords, and such codewords can be searched through probabilistic algorithms. Each row of \mathbf{H}' is a valid codeword of the dual code, so it has at least $A_w \geq (n - k)$ codewords with weight $w \leq d_c m$, where $d_c = n_0 d_v$ is the row weight of \mathbf{H} .

It should be observed that $d_c \ll n$ and the supports of sparse vectors have very small (or null) intersection. So, by introducing an approximation, we can consider $A_w \approx (n - k)$. With similar arguments, and assuming a small m , we can say that the rows of \mathbf{H}' have weight $w \approx d_c m = n_0 d_v m$.

One of the most famous probabilistic algorithms for finding low weight codewords is due to Stern [4] and exploits an iterative procedure. When Stern's algorithm is performed on a code having length n_S and dimension k_S , the probability of finding, in one iteration, one of A_w codewords with weight w is [27]:

$$P_{w, A_w} \leq A_w \cdot \frac{\binom{w}{g} \binom{n_S - w}{k_S/2 - g}}{\binom{n_S}{k_S/2}} \cdot \frac{\binom{w - g}{g} \binom{n_S - k_S/2 - w + g}{k_S/2 - g}}{\binom{n_S - k_S/2}{k_S/2}} \cdot \frac{\binom{n_S - k_S - w + 2g}{l}}{\binom{n_S - k_S}{l}}, \quad (11)$$

where g and l are two parameters whose values must be optimized as functions of the total number of binary operations. So, the average number of iterations needed to find a low weight codeword is $c \geq P_{w, A_w}^{-1}$. Each iteration requires:

$$N = \frac{(n_S - k_S)^3}{2} + k_S(n_S - k_S)^2 + 2gl \binom{k_S/2}{g} + \frac{2g(n_S - k_S) \binom{k_S/2}{g}^2}{2^l} \quad (12)$$

binary operations, so the total work factor is $\text{WF} = cN$.

In the present case, Stern's algorithm is used for attacking the dual of the public code, so $n_S = n$ and $k_S = n - k$. Table 2 reports the values of the maximum work factor achieved (*i.e.*, when $w = d_c m$) by the considered solutions, together with the minimum value of w needed to have work factor $\geq 2^{80}$ (noted by $w(\text{WF} \geq 2^{80})$ in the figure). Based on these results, it seems that all the three systems can be considered secure against attacks to the dual code.

3.2. OTD attacks

In the cryptosystem version proposed in [13], both \mathbf{S} and \mathbf{Q} were chosen sparse, with non-null blocks having row/column weight m , and

$$\mathbf{Q} = \begin{bmatrix} \mathbf{Q}_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{Q}_{n_0-1} \end{bmatrix}. \quad (13)$$

This gave rise to an attack formulated by Otmani, Tillich and Dallot, that is here denoted as *OTD attack* [16].

The rationale of this attack lies in the observation that, by selecting the first k columns of \mathbf{G}' , an eavesdropper can obtain

$$\mathbf{G}'_{\leq k} = \mathbf{S}^{-1} \cdot \begin{bmatrix} \mathbf{Q}_0^{-1} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_1^{-1} & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{Q}_{n_0-2}^{-1} \end{bmatrix}. \quad (14)$$

Then, by inverting $\mathbf{G}'_{\leq k}$ and considering its block at position (i, j) , he can obtain $\mathbf{Q}_i \mathbf{S}_{i,j}$, that corresponds to the polynomial

$$g_{i,j}(x) = q_i(x) \cdot s_{i,j}(x) \bmod (x^p + 1). \quad (15)$$

If both \mathbf{Q}_i and $\mathbf{S}_{i,j}$ are sparse, it is highly probable that $g_{i,j}(x)$ has exactly m^2 non-null coefficients and that its support contains at least one shift $x^{l_a} \cdot q_i(x)$, $0 \leq l_a \leq p-1$.

Three possible strategies have been proposed for implementing this attack. According to the first strategy, the attacker can enumerate all the m -tuples belonging to the support of $g_{i,j}(x)$. Each m -tuple can be then validated through inversion of its corresponding polynomial and multiplication by $g_{i,j}(x)$. If the resulting polynomial has exactly m non-null coefficients, the m -tuple is a shifted version of $q_i(x)$ with very high probability. The second strategy exploits the fact that it is highly probable that the Hadamard product of the polynomial $g_{i,j}(x)$ with a d -shifted version of itself, $g_{i,j}^d(x) * g_{i,j}(x)$, gives a shifted version of $q_i(x)$, for a specific value of d . The eavesdropper can hence calculate all the possible $g_{i,j}^d(x) * g_{i,j}(x)$ and check whether the resulting polynomial has m non null coefficients. As a third strategy, the attacker can consider the i -th row of the inverse of $\mathbf{G}'_{\leq k}$:

$$\mathbf{R}_i = [\mathbf{Q}_i \mathbf{S}_{i,0} | \mathbf{Q}_i \mathbf{S}_{i,1} | \dots | \mathbf{Q}_i \mathbf{S}_{i,n_0-2}]. \quad (16)$$

The linear code generated by

$$\mathbf{G}_{OTD3} = (\mathbf{Q}_i \mathbf{S}_{i,0})^{-1} \cdot \mathbf{R}_i = [\mathbf{I} | \mathbf{S}_{i,0}^{-1} \mathbf{S}_{i,1} | \dots | \mathbf{S}_{i,0}^{-1} \mathbf{S}_{i,n_0-2}] \quad (17)$$

admits an alternative generator matrix:

$$\mathbf{G}'_{OTD3} = \mathbf{S}_{i,0} \mathbf{G}_{OTD3} = [\mathbf{S}_{i,0} | \mathbf{S}_{i,1} | \dots | \mathbf{S}_{i,n_0-2}] \quad (18)$$

that coincides with a block row of matrix \mathbf{S} . When matrix \mathbf{S} is sparse, the code defined by \mathbf{G}'_{OTD3} contains low weight codewords. Such codewords coincide with the rows of \mathbf{G}'_{OTD3} and can be effectively searched through Stern's algorithm.

With the choice of the parameters made in [13], that is almost coincident with the first choice in Table 1, the three OTD attack strategies would require, respectively, $2^{50.3}$, 2^{36} and 2^{32} binary operations. These low values can be easily reached with a standard computer, so that cryptosystem must be considered broken.

However, the OTD attacks rely on the fact that both \mathbf{S} and \mathbf{Q} are sparse and that \mathbf{Q} has block-diagonal form. So, they can be effectively countered by adopting dense \mathbf{S} matrices, without altering the remaining system parameters. With dense \mathbf{S} matrices the eavesdropper cannot obtain \mathbf{Q}_i and $\mathbf{S}_{i,j}$, even knowing $\mathbf{Q}_i\mathbf{S}_{i,j}$, the probability that the support of $g_{i,j}(x)$ contains that of at least one shift of $q_i(x)$ becomes extremely small and the code generated by \mathbf{G}'_{OTD3} does not contain any more low weight codewords.

For preserving the ability of correcting all the intentional errors, it is important that \mathbf{Q} remains sparse (with row/column weight m). The choice of a dense \mathbf{S} influences complexity of the decoding stage, that, however, can be reduced by resorting to efficient computation algorithms for circulant matrices [26].

3.3. Decoding attacks

As stated in the Introduction, the most promising attacks against the McEliece cryptosystem are those aiming at solving the general decoding problem, that is to obtain the error vector \mathbf{e} used for encrypting a ciphertext.

It can be easily shown that \mathbf{e} can be searched as the lowest weight codeword in the extended code generated by

$$\mathbf{G}'' = \begin{bmatrix} \mathbf{G}' \\ \mathbf{x} \end{bmatrix}. \quad (19)$$

In order to evaluate the work factor of such attacks, we refer to Stern's algorithm, whose complexity can be easily evaluated in closed form, as already shown in Section 3.1. Stern's algorithm has been further improved in [5] and, very recently, in [6]. Estimating the work factor of such modified algorithms is more involved, and requires modeling the attack through Markov chains. For this reason, we continue to refer to Stern's original formulation. For our purposes, it seems sufficient to take into consideration that the adoption of optimized algorithms could result in a further speedup of about 12 times, as reported in [6]. According with the expressions reported in Section 3.1, the work factor of a decoding attack against the original McEliece cryptosystem based on Stern's algorithm would be $2^{63.5}$.

In the considered cryptosystem based on QC-LDPC codes, an extra speedup could result by considering the quasi-cyclic nature of the codes. This yields that every block-wise cyclically shifted version of the ciphertext \mathbf{x} is still a valid ciphertext. So, an eavesdropper could continue extending \mathbf{G}'' by adding shifted versions of \mathbf{x} , and could search for as many shifted versions of the error vector. Figure 3 reports the values of the work factor of decoding attacks to the considered cryptosystem as functions of the number of rows added to \mathbf{G}' . The three considered choices of the system parameters reach, respectively, a minimum work factor of $2^{65.6}$, $2^{75.8}$ and $2^{106.5}$ binary operations.

Being the smallest work factors reached by currently known attacks, these values can be considered as the security levels of the three cryptosystems.

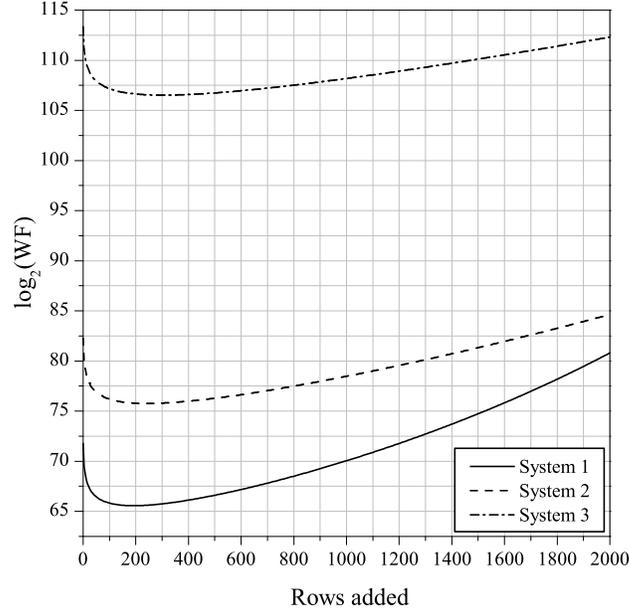


Figure 3. Work factor of decoding attacks based on Stern's algorithm.

4. Complexity

In order to compare the considered cryptosystems with more consolidated solutions, it is important to estimate the complexity of both its encryption and decryption stages.

The encryption complexity is dominated by LDPC encoding, that coincides with calculating the product $\mathbf{u} \cdot \mathbf{G}'$. The number of binary operations needed by such task is denoted as $C_{mul}(\mathbf{u} \cdot \mathbf{G}')$. Further n operations must be considered for addition of the intentional error vector \mathbf{e} . So, the encryption complexity can be expressed as follows:

$$C_{enc} = C_{mul}(\mathbf{u} \cdot \mathbf{G}') + n. \quad (20)$$

The computational cost of matrix multiplication can be reduced by exploiting the fact that each matrix is formed by $p \times p$ binary circulant blocks. Due to the isomorphism with the ring of polynomials $GF(2)[x]/(x^p + 1)$, efficient algorithms for polynomial multiplication over finite fields can be adopted. We refer to the Toom-Cook method, that is very efficient in the cases of our interest, but other strategies are possible [26].

As regards decryption complexity, it can be split into three contributions, corresponding to: i) calculating the product $\mathbf{x} \cdot \mathbf{Q}$, ii) decoding the secret LDPC code and iii) calculating the product $\mathbf{u}' \cdot \mathbf{S}$. So, it can be expressed as follows:

$$C_{dec} = C_{mul}(\mathbf{x} \cdot \mathbf{Q}) + C_{SPA} + C_{mul}(\mathbf{u}' \cdot \mathbf{S}), \quad (21)$$

where C_{SPA} is the number of operations required for LDPC decoding through the sum-product algorithm. By referring to the implementation proposed in [28], we can express C_{SPA} as follows:

Table 3. Parameters of the considered cryptosystems.

	McEliece (1024, 524)	Niederreiter (1024, 524)	RSA 1024-bit mod. public exp. 17	QC-LDPC McEliece 1	QC-LDPC McEliece 2	QC-LDPC McEliece 3
Key Size ^a	67072	32750	256	6144	6144	12288
Rate	0.51	0.57	1	0.75	0.67	0.67
k ^b	524	284	1024	12288	16384	32768
C_{enc}/k ^c	514	50	2402	658	776	1070
C_{dec}/k ^d	5140	7863	738112	4678	8901	12903

^aExpressed in bytes.

^bInformation block length (bits).

^cNumber of binary operations per information bit for encryption.

^dNumber of binary operations per information bit for decryption.

$$C_{SPA} = I_{ave} \cdot n [q (8d_v + 12R - 11) + d_v], \quad (22)$$

where I_{ave} is the average number of decoding iterations and q is the number of quantization bits used inside the decoder (both of them can be estimated through simulations).

By using Eq. (20) and (21), it is possible to estimate the encryption and decryption cost in terms of binary operations per information bit. This has been done in Table 3, that summarizes the main parameters of the considered cryptosystems and compares them with those of more consolidated solutions (for the first three systems the complexity estimates are reported from [5]).

It can be noticed that all the three systems based on QC-LDPC codes have shorter keys and higher rates with respect to the original McEliece cryptosystem and the Niederreiter version; so, they succeed in improving their main drawbacks. In particular, the first QC-LDPC-based system, that reaches a security level comparable with that of the original McEliece cryptosystem, has key size reduced by more than 10 times with respect to it and more than 5 times with respect to the Niederreiter version. Furthermore, the new system has increased transmission rate (up to 3/4).

The security level can be increased at the expenses of the transmission rate: the second QC-LDPC-based system has same key size as the first one, but its transmission rate is reduced from 3/4 to 2/3. As a counterpart, its security level is increased by a factor of about 2^{10} .

Larger keys can be adopted in order to reach higher security levels, that are needed for facing efficient decoding attacks implemented on modern computers. The third QC-LDPC-based system is able to reach a security level of $2^{106.5}$ by doubling the key size (that is still more than 5 times smaller than in the original cryptosystem). It should be noted that the system scales favorably when larger keys are needed, since the key size grows linearly with the code length, due to the quasi-cyclic nature of the codes, while in the original system it grows quadratically.

As concerns complexity, it can be observed that the first QC-LDPC-based cryptosystem has encryption and decryption costs comparable with those of the original McEliece cryptosystem. The Niederreiter version is instead able to significantly reduce the encryption cost. Encryption and decryption complexity increases for the other two QC-LDPC-based variants, but it still remains considerably lower with respect to RSA. On the other hand, RSA has the smallest keys and reaches unitary rate.

5. Conclusion

It has been shown that the adoption of LDPC codes in the framework of the McEliece cryptosystem can help overcoming its drawbacks, that are large keys and low transmission rate. However, such choice must be considered carefully, since the sparse nature of the characteristic matrices of LDPC codes can expose the system to classic as well as newly developed attacks. In particular, the misuse of sparse transformation matrices can expose the system to total break attacks, able to recover the secret key with reasonable complexity.

The adoption of dense transformation matrices permits to avoid such attacks, and the quasi-cyclic nature of the codes still allows to reduce the key size. Furthermore, the McEliece cryptosystem based on QC-LDPC codes can exploit efficient algorithms for polynomial multiplication over finite fields for encryption and low complexity LDPC decoding algorithms for decryption, that reduce its computational complexity.

For these reasons, it seems that the considered variants of the McEliece cryptosystem can be seen as a trade-off between its original version and other widespread solutions, like RSA.

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